One-parameter scaling in weakly disordered one-dimensional systems

1994 J. Phys.: Condens. Matter 6 8259

(http://iopscience.iop.org/0953-8984/6/40/016)

J. Phys.: Condens. Matter 6 (1994) 8259–8268. Printed in the UK

One-parameter scaling in weakly disordered one-dimensional systems

R S Langley

Department of Aeronautics and Astronautics, University of Southampton, Southampton SO9 5NH, UK

Received 26 May 1994, in final form 18 July 1994

Abstract. This work is concerned with the statistical properties of the transmission coefficient t of a 1D disordered system. Previous studies have shown that in the long-length limit the quantity $\ln|t|^{-2}$ has a Gaussian distribution. It has also been shown for a number of specific cases (including the Anderson model, the random phase model, the Gaussian random potential model, and certain structural dynamic systems) that for weak disorder the distribution depends upon a single parameter, in the sense that the variance is equal to twice the mean value. By using a variant of the transfer matrix technique developed by Pendry, it is shown here that this one-parameter scaling is in fact a general property of weakly disordered 1D systems of a certain class, regardless of the details of the adopted model.

1. Introduction

Wave propagation through 1D disordered systems has received much attention in the field of solid state physics, with particular application to electrons on chains of atoms and the associated issue of conductivity. It is perhaps less well known that this problem has also been the subject of much recent interest in the field of structural dynamics (Hodges 1982, Pierre 1990), where the concern is with the effect of manufacturing imperfections on the dynamic performance of a structurally repetitive construction such as a submarine hull or an aircraft fuselage. In recent work concerning 1D elastic waveguides (Langley 1994) it has been found that for weak disorder the elastic wave resistance coefficient $|t|^{-2}$ has the property

$$\mathrm{var}[\ln|t|^{-2}] = 2\langle\ln|t|^{-2}\rangle \tag{1}$$

and further that $\ln|t|^{-2}$ has a Gaussian distribution. This property has also been found for weak disorder in the long-length limit of a number of solid state models, namely the random phase model (Mello 1986), the Gaussian random potential model (Abrikosov 1981, Kumar 1985), and the Anderson model (Pendry 1982b, Stone and Allan 1983, Slevin and Pendry 1990). Here the 'long-length limit' is taken to mean that the system length is much greater than the localization length (Shapiro 1987). Given that the elastic waveguide model and each of the solid state models differ significantly in detail, the question arises as to whether equation (1) is in fact a general result that can be expected for all models of a certain class.

Equation (1) is a specific example of 'scaling' in a disordered system. Early concepts of scaling were expressed in deterministic terms (Abrahams *et al* 1979) although it is now recognized (as reviewed by Shapiro (1986, 1987)) that the statistics of the transmission properties of the disordered system must be considered. In this sense 'scaling' occurs when

0953-8984/94/408259+10\$19.50

the form of the statistical distribution of the transmission coefficient (or an appropriate related variable) is independent of the details of the model under consideration. Furthermore, the system is said to obey n-parameter scaling if the statistical distribution is governed by n independent parameters. Equation (1) states that the mean and the variance of $\ln|t|^{-2}$ are related; since $p(\ln|t|^{-2})$ has been found to be Gaussian, this implies that the statistical distribution of $\ln|t|^{-2}$ is subject to one-parameter scaling, at least under the stated conditions of weak disorder in the long-length limit. Cohen *et al* (1988) have suggested that equation (1) will apply regardless of the details of the 1D model adopted, and the evidence to date would indicate that this is the case. Flores *et al* (1987) and Mello and Shapiro (1988) have shown that the equation is valid for a fairly general 'isotropic uncorrelated' model, in which it is assumed that the transmission and reflection coefficients that describe the properties of an element of the system each have a uniformly distributed phase angle, which is statistically independent of the amplitude. It is shown explicitly in the present work that equation (1) is in fact valid for an even wider class of system: this confirms the concept of one-parameter scaling for this type of system and provides a link between the solid state and the structural dynamics literature.

In what follows, a technique developed by Pendry and his co-workers (Pendry 1982a, Kirkman and Pendry 1984, Slevin and Pendry 1990) is used to calculate the statistical moments of $|t|^{-2}$ for a general 1D system whose properties are described by a 2×2 transfer matrix. The moments are then related to the statistical distribution of $\ln|t|^{-2}$, and this leads to the main conclusion that equation (1) is in fact generally valid for weak disorder in the long-length limit. The analysis also reveals that $\ln|t|^{-2}$ has a Gaussian distribution under these conditions, in agreement with previous work.

2. The statistical moments of $|t|^{-2}$

In what follows the scattering properties of one element of a 1D chain are described in terms of a transfer matrix $\mathbf{T}$. This matrix relates the amplitudes of the right and left travelling waves at the right of the element (a_1' and a_2' say) to those at the left (a_1 and a_2 say) in the form

$$\begin{pmatrix} a_1' \\ a_2' \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}. \tag{2}$$

If the element is conservative and symmetric then $T_{11} = T_{22}^* = 1/t^*$ and $T_{12} = T_{21}^* = -(r/t)^*$ where t and r are the transmission and reflection coefficients, with $|t|^2 + |r|^2 = 1$. It follows from equation (2) that

$$|t|^{-2N} = |a_2'/a_2|^{2N}_{a_1=0} \tag{3}$$

where the notation is such that the term on the right-hand side is to be evaluated under the condition $a_1 = 0$. An expression for the statistical moments of $|t|^{-2}$ for a disordered chain consisting of L elements can now be obtained by following the approach due to Slevin and Pendry (1990). Initially, equation (2) is used to express the product of N right-hand wave amplitudes in the form

$$a_{j_1}' a_{j_2}' \dots a_{j_N}' = \sum_{k_1=1}^{2} \sum_{k_2=1}^{2} \cdots \sum_{k_N=1}^{2} T_{j_1k_1} T_{j_2k_2} \dots T_{j_Nk_N} a_{k_1} a_{k_2} \dots a_{k_N} \tag{4}$$

where each $j_n = 1$ or 2. The product on the left-hand side of this equation may be referenced by the number of twos contained in the set $\{j_1 j_2 \ldots j_N\}$, this number ranging from a minimum of zero to a maximum of N. There are thus $N+1$ distinct Nth-order products of the two right-hand wave amplitudes a'_1 and a'_2. By letting $u'_j = a'_{j_1} a'_{j_2} \ldots a'_{j_N}$, where there are j twos among the subscripts j_n, equation (4) may be re-expressed in the form (Kirkman and Pendry 1984)

$$u'_j = \sum_{k=0}^{N} A_{jk} u_k \tag{5}$$

$$A_{jk} = \sum_{p=0}^{\min(j,k)} {}^{j}C_p \, {}^{N-j}C_{k-p} T_{22}^{p} T_{21}^{j-p} T_{12}^{k-p} T_{11}^{N-j-k+p}. \tag{6}$$

It follows directly from equation (5) that

$$u'_j u'^{*}_m = \sum_{k=0}^{N} \sum_{n=0}^{N} B_{jmkn} u_k u_n^{*} \tag{7}$$

$$\begin{aligned} B_{jmkn} = & \sum_{p=0}^{\min(j,k)} \sum_{q=0}^{\min(m,n)} {}^{j}C_p \, {}^{N-j}C_{k-p} \, {}^{m}C_q \, {}^{N-m}C_{n-q} T_{11}^{N-j-k+p+q} \\ & \times T_{11}^{*(N-m-n+p+q)} T_{12}^{k+m-p-q} T_{12}^{*(j+n-p-q)} \end{aligned} \tag{8}$$

where use has been made of the fact that $T_{22} = T_{11}^{*}$ and $T_{21} = T_{12}^{*}$. Equation (7) may be expressed in matrix form by introducing the indexing notation

$$v'_\alpha = u'_j u'^{*}_m \tag{9}$$

$$\alpha = j(N+1) + m + 1 \tag{10}$$

$$v_\beta = u_k u_n^{*} \tag{11}$$

$$\beta = k(N+1) + n + 1 \tag{12}$$

which yields

$$v' = \mathbf{C} v \tag{13}$$

where $C_{\alpha\beta} = B_{jmkn}$ has dimension $(N+1)^2 \times (N+1)^2$. Equation (13) relates to wave transmission across a single element of a 1D chain—clearly the matrix $\mathbf{C}$ plays the role of a transfer matrix, and the corresponding result for wave transmission across L sequential elements will have the form

$$v' = \left\{ \prod_{r=1}^{L} \mathbf{C}_r \right\} v \tag{14}$$

where $\mathbf{C}_r$ is the transfer matrix for the rth element. It follows from equation (3) and the definition of v and v' that the final diagonal entry of the matrix that appears in equation (14) is just $|t_L|^{-2N}$, where t_L is the transmission coefficient of the chain. If the disorder in the element properties is taken to be statistically homogeneous, then the average of this term is

$$\langle |t_L|^{-2N} \rangle = \boldsymbol{w}^{\mathrm{T}} \left\langle \prod_{r=1}^{L} \mathbf{C}_r \right\rangle \boldsymbol{w} = \boldsymbol{w}^{\mathrm{T}} \langle \mathbf{C} \rangle^L \boldsymbol{w} \tag{15}$$

where $\boldsymbol{w}$ is a vector of dimension $(N+1)^2$ whose entries are all zero apart from the final component, which is unity. This result may be re-expressed in the form

$$\langle |t_L|^{-2N} \rangle = \sum_{j=1}^{(N+1)^2} \lambda_j^L e_j e_j' \tag{16}$$

where λ_j is the jth eigenvalue of $\langle \mathbf{C} \rangle$, and e_j and e_j' are the final entries in the jth left- and right-hand eigenvectors respectively. In the long-length limit ($L \to \infty$) the summation that appears in equation (16) will be dominated by the eigenvalue of largest modulus, $\lambda_{\max}$ say, and thus

$$\langle |t_L|^{-2N} \rangle \simeq \lambda_{\max}^L e_{\max} e_{\max}'. \tag{17}$$

For the ordered system it can be shown that $\lambda_{\max} = 1$; if the effect of disorder produces a change in $\lambda_{\max}$ of order Δ, then equation (17) will only be a valid approximation if $\Delta L \gg 1$. In the notation of Shapiro (1987) this is equivalent to the condition $L/L_c \gg 1$ where L_c is the 'localization length' of the system. This condition is assumed to be met in what follows. It can be noted that the result given by equation (17) is basis dependent, in the sense that the basis used to describe the wave amplitudes a_1 and a_2 that appear in equation (2) will affect the value of $e_{\max} e_{\max}'$ (but not $\lambda_{\max}$). Thus a linear transformation to a new set of wave coordinates, b_1 and b_2 say, ($\boldsymbol{b} = \mathbf{S}\boldsymbol{a}$ for some matrix $\mathbf{S}$) will change the statistical moments of $|t_L|^{-2}$. This is not true of the logarithm of the statistical moments however, since for large L equation (17) yields

$$\ln \langle |t_L|^{-2N} \rangle \simeq L \ln \lambda_{\max}. \tag{18}$$

The relationship between this result and the statistical distribution of $\ln |t_L|^{-2}$ is discussed in the following section.

3. The relation to the statistical distribution of $\ln|t_L|^{-2}$

Slevin and Pendry (1990) have shown that in the long-length limit the variable $\ln |t_L|^{-2}$ becomes normally distributed. More generally, if $\ln |t_L|^{-2}$ is assumed to have a near-normal distribution then its probability density function may be written in the form of a Gram–Charlier or Edgeworth series (Ibrahim 1985) as follows:

$$p(z) = (1/\sqrt{2\pi}) \mathrm{e}^{-z^2/2} \sum_{n=0}^{\infty} h_n H_n(z) \tag{19}$$

$$H_n(z) = (-1)^n \mathrm{e}^{z^2/2} \frac{\mathrm{d}^n}{\mathrm{d}z^n} \mathrm{e}^{-z^2/2} \tag{20}$$

$$z = (\ln |t_L|^{-2} - c_1)/c_2^{1/2} \tag{21}$$

$$c_1 = \langle \ln |t_L|^{-2} \rangle \tag{22}$$

$$c_2 = \mathrm{var}[\ln |t_L|^{-2}]. \tag{23}$$

Here $H_n(z)$ is the nth Hermite polynomial and c_1 and c_2 are the first two cumulants of $\ln|t_L|^{-2}$—that is, the mean and variance. Furthermore, the coefficients h_n that appear in equation (19) are dependent on the higher-order cumulants—for the Gram–Charlier series the first terms are

$$h_0 = 1 \tag{24}$$

$$h_1 = h_2 = 0 \tag{25}$$

$$h_3 = c_3/6c_2^{3/2} \tag{26}$$

$$h_4 = c_4/24c_2^2 \tag{27}$$

where c_n represents the nth cumulant so that c_3 is the skewness and c_4 the kurtosis. It follows from equation (19) that the statistical moments of $|t_L|^{-2}$ can be expressed in the form

$$\langle|t_L|^{-2N}\rangle = \exp(Nc_1 + N^2c_2/2)\sum_{n=0}^{\infty} h_n c_2^{n/2} N^n \tag{28}$$

which implies that

$$\ln\langle|t_L|^{-2N}\rangle = Nc_1 + N^2c_2/2 + \ln[1 + h_3c_2^{3/2}N^3 + h_4c_2^2N^4 + \cdots]. \tag{29}$$

It can be noted that the grouping of terms which appears in this result arises from the Gram–Charlier representation, equation (19). A more direct analysis leads to equation (29) with the right-hand side replaced by $\Sigma c_n N^n/n!$; this is in fact just a re-ordering of the terms that appear in the present representation. It is clear from equation (29) that knowledge of $\ln\langle|t_L|^{-2}\rangle$ will give some insight into the nature of the cumulants of $\ln|t_L|^{-2}$—this issue is further considered in the following section for the case of weak disorder.

4. Weak disorder

As mentioned in section 2, the wave amplitudes a_1 and a_2 that appear in equation (2) can be expressed in any basis. In the present analysis it is convenient to choose that basis for which the transfer matrix $\mathbf{T}$ is diagonal in the absence of disorder. Before considering the effects of disorder, it is useful to consider the structure of the matrix $\mathbf{C}$ that appears in equation (13) for the case of a perfect system: since $T_{12} = 0$ for the chosen basis, it can readily be shown that $\mathbf{C}$ is diagonal. Furthermore, the α diagonal entry has the form

$$C_{\alpha\alpha} = T_{11}^{N-j+m} T_{11}^{*(N-m+j)} \tag{30}$$

where $\alpha = j(N+1)+m+1$, in accordance with equation (10). Within an energy band T_{11} has the form $T_{11} = \mathrm{e}^{\mathrm{i}\epsilon}$ where ϵ is the Bloch wavenumber (usually known as the propagation constant in structural dynamics), and thus $C_{\alpha\alpha}$ will be unity whenever $j = m$. Since j and m range from zero to N there will be $N+1$ unit diagonals of this type, which implies that $\mathbf{C}$ will have $N+1$ unit eigenvalues. Other unit eigenvalues can occur if ϵ is a rational fraction of π; for example, at mid-band $\epsilon = \pi/2$ and additional unit eigenvalues will be associated with the terms $m - j = \pm 2r$ for any integer r—this leads to a well known anomaly in the localization length at mid-band (Kappus and Wegner 1981). In general, if $\epsilon = n_1\pi/n_2$ then additional unit eigenvalues will arise for $N \geqslant n_2$, and the statistical moments of $|t_L|^{-2}$ from n_2 onwards will be affected. Such effects are not considered in the present analysis, where it is assumed from the outset that ϵ is not a rational fraction of π.

The $N+1$ unit eigenvalues of $\mathbf{C}$ in the absence of disorder correspond to the case $m = j$ in equation (30); in more detail, the matrix will have $2N+1$ distinct eigenvalues corresponding to $m - j = r$ say ($-N \leqslant r \leqslant N$), with the rth eigenvalue being repeated $N+1-r$ times. For weak disorder the effect of irregularity on the eigenvalues of $\mathbf{C}$ can be investigated by considering each of the $2N+1$ blocks of repeated eigenvalues independently. For the rth eigenvalue this procedure consists of analysing the effect of disorder on an $(N+1-r)\times(N+1-r)$ sub-matrix formed from the entries of $\mathbf{C}$ that have $m - j = n - k = r$, so that $\alpha = j(N+2)+r+1$ and $\beta = k(N+2)+r+1$. It follows from equation (8) that a general term in this sub-matrix (C'_{jk} say) has the form

$$C'_{jk} = \sum_{p=0}^{\min(j,k)} \sum_{q=0}^{\min(j+r,k+r)} {}^{j}C_p\,{}^{N-j}C_{k-p}\,{}^{j+r}C_q\,{}^{N-j-r}C_{k+r-q}|T_{11}|^{2(N-j-k-r+p+q)} \times \cdots |T_{12}|^{2(j+k+r-p-q)}(T_{11}/T_{11}^*)^r. \tag{31}$$

As mentioned previously, in the absence of disorder the entries of the transfer matrix have the form $T_{11} = \mathrm{e}^{\mathrm{i}\epsilon}$ and $T_{12} = 0$. In general, disorder will be caused by random variations in one or more of the basic physical properties of the system. If the disorder is considered to arise from variations in a single parameter, θ say, then a Taylor series expansion of the transfer matrix will yield

$$T_{11} = \mathrm{e}^{\mathrm{i}\epsilon}(1 + f_1\theta + f_2\theta^2 + \cdots) \tag{32}$$

$$T_{12} = g_1\theta + g_2\theta^2 + g_3\theta^3 + \cdots \tag{33}$$

where f_i and g_i are the appropriate coefficients and, without loss of generality, θ can be considered to have zero mean. The fact that the determinant of the transfer matrix must always be unity implies that $|T_{11}|^2 - |T_{12}|^2 = 1$, which means that there are certain relationships between the coefficients f_i and g_i—furthermore, it follows that f_1 must be purely imaginary since $|T_{11}|^2$ cannot contain a first-order term in θ. Equations (32) and (33) then lead to the following result:

$$|T_{11}|^2 = 1 + \Delta \tag{34}$$

$$|T_{12}|^2 = \Delta \tag{35}$$

$$\Delta = d_2\theta^2 + d_3\theta^3 + \cdots \tag{36}$$

where the coefficients d_i may be expressed in terms of g_i. Now, to second order in θ, the term $(T_{11}/T_{11}^*)^r$ that appears in equation (31) may be written in the form

$$\begin{aligned}(T_{11}/T_{11}^*)^r &= \mathrm{e}^{2\mathrm{i}\epsilon r}\{1+\theta r(f_1 - f_1^*)+\theta^2[rf_2 - rf_2^* + r(r-1)f_1^2/2 + r(r+1)f_1^{*2}/2 - r^2|f_1|^2]\} \\ &= \mathrm{e}^{2\mathrm{i}\epsilon r}(1+\delta)\end{aligned} \tag{37}$$

where δ is defined accordingly. Given that θ has zero mean, it follows from equation (36) that for small disorder the condition $\langle\Delta^2\rangle \ll \langle\Delta\rangle$ will normally apply, which means that the effect of weak disorder on $\langle \mathbf{C}' \rangle$ may be investigated by retaining only those terms up to first order in Δ in equation (31). Furthermore, it follows from equations (36) and (37) that terms of the form $\delta\Delta^n$ with $n \geqslant 1$ may also be neglected. Although the present development is based on a single physical parameter θ, similar conclusions regarding the order of the terms δ and Δ would arise from a more general multi-parameter expansion of T_{11} and T_{12} in place of equations (32) and (33).

An approximate solution for the eigenvalues of the matrix $\mathbf{C}'$ and hence $\langle \mathbf{C}' \rangle$ can now be obtained by retaining in equation (31) only those terms that are zero order or first order in Δ and δ. The first group of such terms arises for $j+k+r-p-q=0$, so that $|T_{12}|^2$ does not appear in the summation: this condition is only possible for $j=k=p=q-r$, and the resulting contribution to $\mathbf{C}'$ consists of the term $\mathrm{e}^{2\mathrm{i}\epsilon r}(1+\Delta)^N(1+\delta) \simeq \mathrm{e}^{2\mathrm{i}\epsilon r}(1+N\Delta+\delta)$ on each diagonal. The second group of non-negligible terms has $j+k+r-p-q=1$, so that the power of $|T_{12}|^2$ in equation (31) is unity. There are three situations for which this condition can arise, namely (i) $k=p=q-r=j-1$, (ii) $k=p+1=q+1-r=j+1$, and (iii) $k=j$ with $j=p=q+1-r$ or $j=p+1=q-r$. Case (i) leads to the presence of the term $\mathrm{e}^{2\mathrm{i}\epsilon r}j(j+r)\Delta$ in $C'_{j,j-1}$, while case (ii) leads to $\mathrm{e}^{2\mathrm{i}\epsilon r}(N-j)(N-j-r)\Delta$ in $C'_{j,j+1}$. Finally, case (iii) adds the two terms $\mathrm{e}^{2\mathrm{i}\epsilon r}j(N-j)\Delta$ and $\mathrm{e}^{2\mathrm{i}\epsilon r}(j+r)(N-j-r)\Delta$ to C'_{jj}. It thus follows that to first order in Δ and δ, $\mathbf{C}'$ is a tridiagonal matrix with

$$C'_{j,j-1} = j(j+r)\Delta\mathrm{e}^{2\mathrm{i}\epsilon r} \tag{38}$$

$$C'_{jj} = \{1+N\Delta+\delta+(j+r)(N-r-j)\Delta+j(N-j)\Delta\}\mathrm{e}^{2\mathrm{i}\epsilon r} \tag{39}$$

$$C'_{j,j+1} = (N-j)(N-j-r)\Delta\mathrm{e}^{2\mathrm{i}\epsilon r}. \tag{40}$$

It is readily confirmed that each row of this matrix sums to a common result, μ say, where

$$\mu = \sum_{k=1}^{3} C'_{j,j+k-2} = 1+[N(N+1)-r^2]\Delta+\delta. \tag{41}$$

It follows that μ is an eigenvalue of $\mathbf{C}'$, with the associated right eigenvector having each entry equal to unity. What is actually required for the purposes of the present analysis is the largest eigenvalue of the matrix $\langle \mathbf{C} \rangle$, so that the statistical moments of $|t_L|^{-2}$ may be estimated from equation (18). The result given by equation (41) is by contrast one of the eigenvalues of the rth sub-block of $\mathbf{C}$; the corresponding eigenvalue for the rth sub-block of $\langle \mathbf{C} \rangle$ is simply given by equation (41) with Δ and δ replaced by $\langle \Delta \rangle$ and $\langle \delta \rangle$ respectively, which in fact just yields $\langle \mu \rangle$. It can be noted from equations (34)–(37) that while $\langle \Delta \rangle$ is real, $\langle \delta \rangle$ may be complex. However, it also follows from equation (37) that both the real and imaginary components of $\langle \delta \rangle$ are of order $\langle \Delta \rangle$, which means that the imaginary part of $\langle \delta \rangle$ makes a negligible contribution to the modulus of $\langle \mu \rangle$. Furthermore, to first order in $\langle \Delta \rangle$ the norm of $\langle \mathbf{C}' \rangle$ is given by $\|\langle \mathbf{C}' \rangle\|_\infty = |\langle \mu \rangle|$, which implies that $\langle \mu \rangle$ is the eigenvalue of largest modulus for the rth sub-block since it is known that $\|\langle \mathbf{C}' \rangle\|_\infty \geqslant |\lambda_j|$ for any eigenvalue λ_j. The value of r that yields the maximum value of $\langle \mu \rangle$ will depend upon the form of $\langle \delta \rangle$: for the Anderson model it can be shown that $\langle \delta \rangle = -2r^2\langle \Delta \rangle$, while more generally equation (37) implies that the real part of $\langle \delta \rangle$ will be negative while the imaginary part has a negligible effect. It thus follows from equation (41) that the largest eigenvalue of $\langle \mathbf{C} \rangle$ corresponds to the case $r=0$, and can thus be written in the form

$$\lambda_{\max} = 1+N(N+1)\langle \Delta \rangle \tag{42}$$

$$\ln \lambda_{\max} \simeq N(N+1)\langle \Delta \rangle. \tag{43}$$

A comparison between equations (18), (29) and (43) then reveals that for weak disorder the first two cumulants of $\ln|t_L|^{-2}$ have the form

$$c_2 = 2c_1 = 2L\langle\Delta\rangle. \tag{44}$$

This result confirms the validity of equation (1) for a weakly disordered 1D system of the type described by equation (2). It also follows that to the present level of approximation $\ln|t_L|^{-2}$ has a Gaussian distribution, in the sense that the higher-order cumulants are either independent of L or are of second or higher order in $\langle\Delta\rangle$.

5. Example applications

5.1. The Anderson model

Slevin and Pendry (1990) have shown that the transfer matrix $\mathbf{T}$ for the Anderson model with diagonal disorder has the form

$$\mathbf{T} = \begin{pmatrix} (1-\mathrm{i}\delta_n)\mathrm{e}^{\mathrm{i}k} & -\mathrm{i}\delta_n\mathrm{e}^{\mathrm{i}k} \\ \mathrm{i}\delta_n\mathrm{e}^{-\mathrm{i}k} & (1+\mathrm{i}\delta_n)\mathrm{e}^{-\mathrm{i}k} \end{pmatrix} \tag{45}$$

where the wavenumber k is related to the energy E by

$$E = -2\cos k. \tag{46}$$

The disorder parameter δ_n that appears in equation (45) is written in the form

$$\delta_n = \epsilon_n/2\sin k \tag{47}$$

where ϵ_n is the randomly distributed site energy, with $\langle\epsilon\rangle = 0$. In the absence of disorder ($\delta_n = 0$), the matrix $\mathbf{T}$ is diagonal, and thus the basis used to describe the wave motion conforms with the analysis of section 4. It then follows from equations (34) and (44) that

$$c_2 = 2c_1 = 2L\mathrm{var}(\epsilon_n)/(4-E^2) \tag{48}$$

which is in full agreement with previous work. It can be noted that Slevin and Pendry (1990) have computed c_1 and c_2 by analytical continuation of the analysis of section 2 to non-integer N. This approach leads to an expression for c_n in terms of the nth derivative with respect to N of the largest eigenvalue of $\langle\mathbf{C}\rangle$; for c_1 an analytical result was obtained, whereas numerical methods were employed for c_2. Although the numerical procedure did not converge for higher values of c_n it was conjectured that $c_n = 0$ for $n > 2$. This conjecture is confirmed by the present analytical approach. It can further be noted that Pendry (1982b) has obtained a similar result to equation (40) for the Anderson model by application of the symmetric group.

5.2. The structural waveguide

Many structural configurations consist of a series of discontinuities nominally evenly spaced on an otherwise uniform structure. A prime example is that of a ring stiffened cylindrical shell, as used in an aircraft fuselage or a submarine hull, where the rings or frames form the discontinuities and irregular frame spacing provides the source of the disorder. For each circumferential Fourier component of the shell displacement, the structure may be modelled as a one-dimensional waveguide with the transfer matrix (Langley 1994)

$$\mathbf{T} = \begin{pmatrix} e^{-ikl}/t^* & -e^{-ikl}(r/t)^* \\ -e^{ikl}(r/t) & e^{ikl}/t \end{pmatrix}. \tag{49}$$

Here k is the structural wavenumber, l is the spacing between the frames, and t and r are the frame transmission and reflection coefficients. Numerical studies (Langley 1994) of the eigenvalues of the matrix $\langle \mathbf{C}(N) \rangle$ for this case have revealed that for weak disorder in l

$$\lambda_{\max}(N) = [\lambda_{\max}(1)]^{N(N+1)/2} \tag{50}$$

which conforms to equation (42) for small Δ. Now it follows from equations (42) and (44) that the value of c_1 may be deduced from knowledge of $\lambda_{\max}(1)$: in this case $\lambda_{\max}(1)$ may be evaluated by the perturbation technique to yield (just off mid-band)

$$c_1 = 2|r/t|^2(k\sigma)^2 L \tag{51}$$

where σ is the standard deviation of l, and L is the number of frames. This result has been found to agree with simulation studies for a beam on multiple simple supports (Langley 1994, Bouzit and Pierre 1992), thus confirming the validity of the analysis of section 4 for the system governed by equation (49).

6. Conclusions

It has been shown that equation (1) is valid in the long-length limit ($L/L_c \gg 1$ in the notation of Shapiro (1987)) for any weakly disordered 1D system that may be represented in the form of equation (2); this result has previously been demonstrated only for a number of specific models (including the Anderson model, the random phase model, and the Gaussian random potential model). The present analysis has also confirmed that $\ln|t_L|^{-2}$ has a Gaussian distribution under these conditions. It has thus been demonstrated that the variable $\ln|t_L|^{-2}$ obeys one-parameter scaling under the stated conditions, in the sense that the form of the statistical distribution $p(\ln|t_L|^{-2})$ is independent of the detailed model considered and is described by a single parameter. Much of the earlier work on scaling has been summarized by Cohen *et al* (1988)—it is suggested therein that equation (1) may be of general validity, and this has been demonstrated explicitly by the present analysis. While being based on the analysis technique developed by Pendry (1982a), the present approach is relatively direct in that the use of analytic continuation (Slevin and Pendry 1990) and the symmetric group (Pendry 1982b) has been avoided. This has been made possible by limiting the present discussion to the case of weak disorder.

In the case of strong disorder, Slevin and Pendry (1990) have shown that while $\ln|t_L|^{-2}$ remains Gaussian in the long-length limit, equation (1) no longer holds; this behaviour has recently been confirmed numerically for a disordered structural waveguide (Langley 1994). More generally, Cohen *et al* (1988) have shown that two-parameter scaling may generally be expected for strong disorder and high-dimensional systems.

References

Abrahams E, Anderson P W, Licciardello D C and Ramakrishnan T V 1979 Scaling theory of localization: absence of quantum diffusion in two-dimensions *Phys. Rev. Lett.* **42** 673–6

Abrikosov A A 1981 The paradox with the static conductivity of a one-dimensional metal *Solid State Commun.* **37** 997–1000

Bouzit D and Pierre C 1992 Vibration confinement phenomena in disordered, mono-coupled, multi-span beams *J. Vib. Acoust.* **114** 521–30

Cohen A, Roth Y and Shapiro B 1988 Universal distributions and scaling in disordered systems *Phys. Rev.* B **38** 12 125–32

Flores J, Mello P A and Mansivais G 1987 Statistical distribution of the resistance in a one-dimensional disordered electrified chain *Phys. Rev.* B **35** 2144–50

Hodges C H 1982 Confinement of vibration by structural irregularity *J. Sound Vib.* **82** 411–24

Ibrahim R A 1985 *Parametric Random Vibration* (Letchworth: Research Studies)

Kappus M and Wegner F 1981 Anomaly in the band centre of the one-dimensional Anderson model *Z. Phys.* B **45** 15–21

Kirkman P D and Pendry J B 1984 The statistics of one-dimensional resistances *J. Phys. C: Solid State Phys.* **17** 4327–44

Kumar N 1985 Resistance fluctuation in a one-dimensional conductor with static disorder *Phys. Rev.* B **31** 5513–15

Langley R S 1994 Wave transmission through one-dimensional near-periodic structures: optimum and random disorder *J. Sound Vib.* submitted

Mello P 1986 Central-limit theorem on groups *J. Math. Phys.* **27** 2876–91

Mello P and Shapiro B 1988 Existence of a limiting distribution for disordered electronic conductors *Phys. Rev.* B **37** 5860–3

Pendry J B 1982a The evolution of waves in disordered media *J. Phys. C: Solid State Phys.* **15** 3493–511

—— 1982b 1D localisation and the symmetric group *J. Phys. C: Solid State Phys.* **15** 4821–34

Pierre C 1990 Weak and strong vibration localization in disordered structures: a statistical investigation *J. Sound Vib.* **139** 111–32

Shapiro B 1986 Probability distributions in the scaling theory of localization *Phys. Rev.* B **34** 4394–7

—— 1987 Scaling properties of probability distributions in disordered systems *Phil. Mag.* B **56** 1031–44

Slevin K M and Pendry J B 1990 Statistics and scaling in one-dimensional disordered systems *J. Phys.: Condens. Matter* **2** 2821–32

Stone A D and Allan D C 1983 Phase randomness in the one-dimensional Anderson model *Phys. Rev.* B **27** 836–43